Calculus (1)

What is calculus ?

Calculus is the study of how things change. It provide a framework for modeling systems in which there is change, and away to deduce the predictions of such models.

Ex1 Average velocity during the first 3 seconds ?

calculus is Divided into two categories

Differential calculus
(Rate of change)

integral calculus
(Accumulation)

Fundamental Theorem of calculs
(connects Differential and integral calculus)

Real Numbers

1) Natural numbers are the counting numbers $\{1, 2, 3, ---\}$ or the whole numbers $\{0, 1, 2, 3, ---\}$, is denoted by $\mathbb{N}$

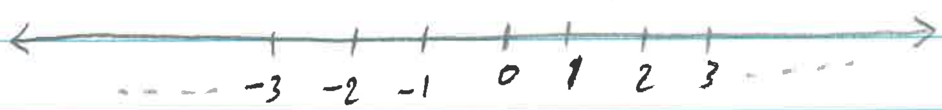
2) The numbers $\{----, -3, -2, -1, 0, 1, 2, 3, ----\}$ are called integers. The set of all integers is denoted by $\mathbb{Z}$

3) Rational numbers in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$, that is $\mathbb{Q} = \{ \frac{p}{q} ; p, q \text{ are integers and } q \neq 0 \}$

4) Rational numbers together with irrational numbers are called real numbers. The set of all real numbers is denoted by $\mathbb{R}$

*Note that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

* Real numbers can be represented by points on a line, called the real number line.



Relation; The following nine types of subsets of $\mathbb{R}$ are called interval.

$[a,b] = \{x \in \mathbb{R} ; a \leq x \leq b\}$

$(a,b] = \{x \in \mathbb{R} ; a < x \leq b\}$

$(a,b) = \{x \in \mathbb{R} ; a < x < b\}$

$[a,\infty) = \{x \in \mathbb{R} ; x \geq a\}$

$(a,\infty) = \{x \in \mathbb{R} ; x > a\}$

$(-\infty, b] = \{x \in \mathbb{R} ; x \leq b\}$

$(-\infty, b) = \{x \in \mathbb{R} ; x < b\}$

$(-\infty, \infty) = \mathbb{R}$

where a and b are real numbers with $a<b$ and ∞ and $-\infty$ (real infinity) and (minus infinity) are just symbols but not real numbers.

- Terminology

1) The intervals in the form (a,b), $[a,b]$, $(a,b]$ and $[a,b]$ are called bounded interval and these in the form $(-\infty,b)$, $(-\infty,b]$, (a,∞), $[a,\infty)$ and $(-\infty,\infty)$ are called unbounded intervals

2- Intervals in the form (a,b), $(-\infty,b)$, (a,∞) and $(-\infty,\infty)$ are called open intervals.

3- Intervals in the form $[a,b]$, $(-\infty,b]$, $[a,\infty)$ and $(-\infty,\infty)$ are called closed intervals.

4- Intervals in the form $[a,b]$ are called closed and bounded intervales.

Ex Let 1) $A=[1,5]$ and $B=(3,10]$

2) $A=[-3,3]$ and $B=(9,11)$

for each part determine 1) whether if it is an open intervals, a closed interval

2) Find $A \cap B$ and determine whether it is an interval.

3) Find $A \cup B$ and determine whether it is an interval

Sol 1) A and B are not open intervals

A is closed interval but B is not closed interval

Inequalities

1) if $a<b$ and $b<c$ then $a<c$
2) if $a<b \Rightarrow a+c<b+c$ and $a-c<b-c$
3) if $a<b \Rightarrow ac<bc$ when $c>0$
   and $ac>bc$ when $c<0$

4) if $a<b$ & $c<d \Rightarrow a+c<b+d$

5) if $a>0$ & $b>0$ or $a<0$ & $b<0$ and $a<b$
$\Rightarrow \frac{1}{a} > \frac{1}{b}$

EX1 Solve $3+7x \leq 2x-9$

Sol1 $3+7x \leq 2x-9$ $\quad -3$

$7x \leq 2x-12$ $\quad -2x$

$5x \leq -12$

$x \leq \frac{-12}{5}$

EX1 Solve $7 \le 2-5x < 9$

Sol $7 \le 2-5x$ و $2-5x < 9$

$7 \le 2-5x < 9 \qquad -2$

$5 \le -5x < 7 \qquad \div 5$

$-1 \ge x > \frac{-7}{5}$

$\frac{-7}{5} < x \le -1$

EX1 Solve $x^2-3x > 10$

Sol $x^2-3x-10 > 0$

$(x+2)(x-5) > 0$

$x+2=0$ أو $x-5=0$

$\Rightarrow x=-2$ أو $x=5$

$(-\infty,-2)$, $(-2,5)$, $(5,\infty)$

$(-\infty,-2) \cup (5,\infty)$

Ex1 Solve $\frac{2x-5}{x-2} < 1$

1) first find the zeroes from the numerator and the undefined points from the denominator

Sol $\frac{2x-5}{x-2} < 1 \Rightarrow \frac{2x-5}{x-2} - 1 < 0$

2) divide the number line into intervales

$$\Rightarrow \frac{2x-5-x+2}{x-2} < 0$$

$$\Rightarrow \frac{x-3}{x-2} < 0 \Rightarrow x-3=0 \Rightarrow x=3$$

$$x-2=0 \Rightarrow x=2$$

$+ \quad \circ \quad - \quad \circ \quad +$

$\quad\;\; 2 \qquad\;\; 3$

$(-\infty, 2)$, $(2,3)$, $(3,\infty) \Rightarrow 2 < x < 3$

Absolute Value:

Def: The absolute value of real number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Ex1 $|3| = 3$, $\left|-\frac{2}{5}\right| = \frac{2}{5}$, $|0| = 0$

Ex! Solve $|x-3| = 4$

Sol! $x-3=4 \Rightarrow x=7$

$x-3=-4 \Rightarrow x=-1$

Ex! Solve $|3x-2| = |5x+4|$

Sol! $3x-2 = 5x+4 \Longrightarrow 3x-5x = 4+2 \Rightarrow x = \frac{6}{-2} = -3$

$3x-2 = -(5x+4) \Rightarrow 3x+5x = -4+2 \longrightarrow x = \frac{-2}{8} = \frac{-1}{4}$

Thm! For any a & b real numbers

1) $\sqrt{a^2} = |a|$
2) $|-a| = |a|$
3) $|ab| = |a||b|$
4) $|a/b| = |a|/|b|$
5) $|a+b| \leq |a| + |b|$

Def! The distance between two points a and b on a number line we use the distance formula:

$d = |b-a| = |a-b|$

Remark:

1) $|b-a|$ is the distance between b and a

2) $|b+a|$ is the distance between b and $-a$ $\left(\text{since } |x+a| = |x-(-a)|\right)$

3) $|a|$ is the distance between x and the origin (since $|x| = |x-0|$)

4) For $k > 0$ then

i) $|b-a| < k \Rightarrow -k < b-a < k \Rightarrow a-k < b < a+k$

ii) $|b-a| > k \Rightarrow b-a < -k$ or $b-a > k$

$b < a-k$ or $b > a+k$

Ex1 Solve (a) $|x-4| < 5$ b) $|x+5| \geq 3$

c) $\dfrac{1}{|2x-5|} > 7$

Sol1 (a) $|x-4| < 5 \Rightarrow -5 < x-4 < 5$

$\Rightarrow -1 < x < 9$

b) $|x+5| \geq 3 \Rightarrow$ i) $x+5 \leq -3$ ii) $x+5 \geq 3$ $\Rightarrow$ $x \leq -8$, $x \geq -2$ $\Rightarrow (-\infty, -8] \cup [-2, \infty)$

عامل مشترك

c) $\dfrac{1}{|2x-5|} > 7 \Rightarrow |2x-5| < \dfrac{1}{7} \Rightarrow |2|\,|x-\frac{5}{2}| < \dfrac{1}{7}$

$\Rightarrow |x-\frac{5}{2}| < \frac{1}{14}$

$\Rightarrow \frac{-1}{14} < x-\frac{5}{2} < \frac{1}{14}$

$\Rightarrow -\frac{1}{14} + \frac{5}{2} < x < \frac{1}{14} + \frac{5}{2}$

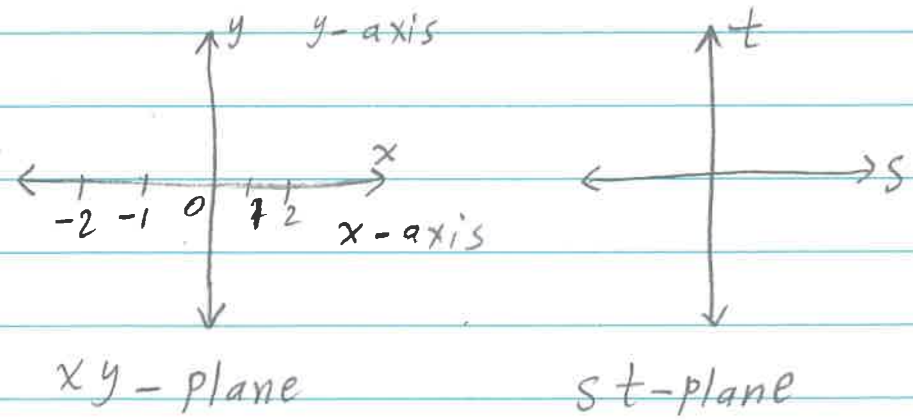
Cartesian coordinate system :-   المستوى الديكارتي

A cartesian coordinate system consist of two parpendicular coordinate lines , called coordinate axes.

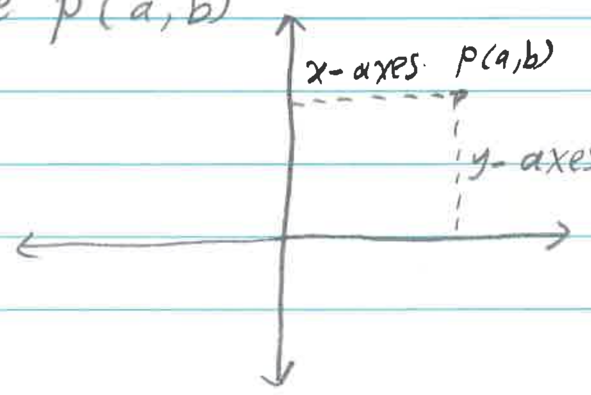
- The intersection of the axes is called the origin of the coordinate system.

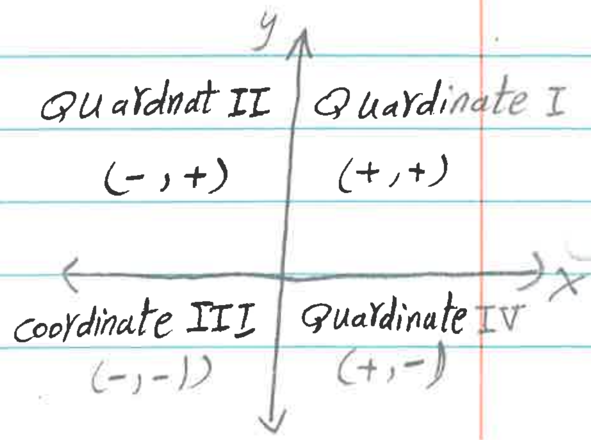
- The horizontal axis is called the x-axis and the vertical axis is called the y-axis

- The plane of the axes together is called the xy-plane



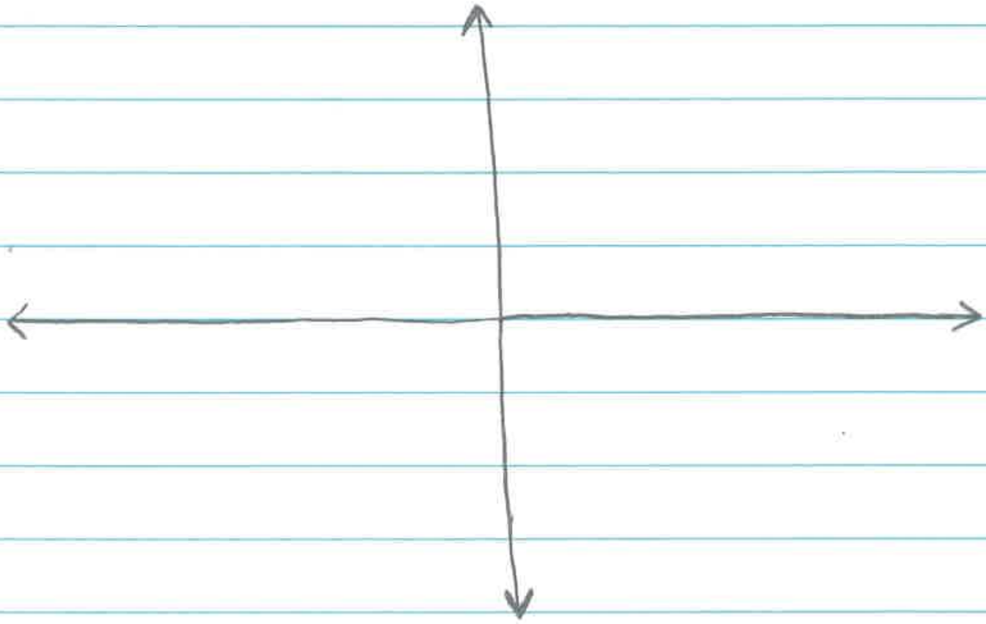
- Every point p in a coordinate plane can be associated with a unique orderd pair of real numbers by drawing two lines through p, and we will say p has coordinates (a,b) and write p(a,b)





Ex1 Plot the following points:

$P(3,5)$, $Q(-2,3)$, $R(-4,-2)$ and $S(2,-2)$

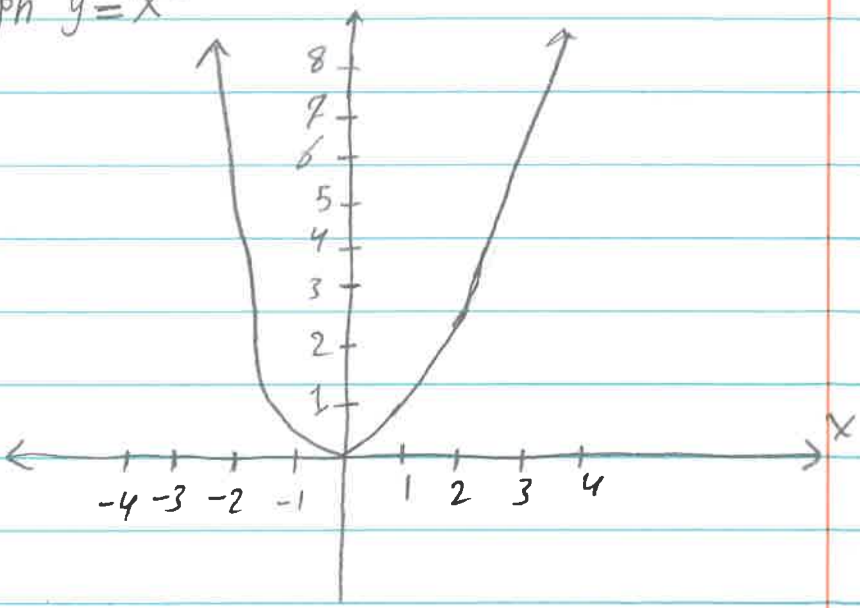


Def:- The set of all solutions of an equation in x and y is called the solution set of the equation, and the set of all points in the xy-plane whose coordinates are members of the solution set is called the graph of the equations.

Ex1 Sketch the graph $y=x^2$

Sol1

x	
0	0
1	1
2	4
3	9
-1	1
-2	4
-3	9



EX1 Sketch the graph $y=\sqrt{x}$

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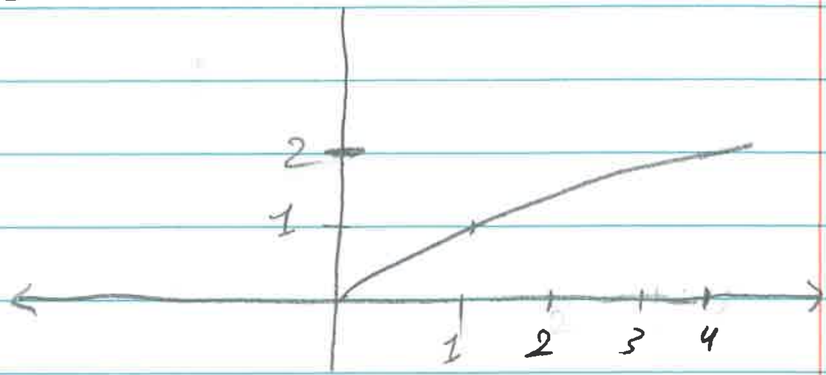
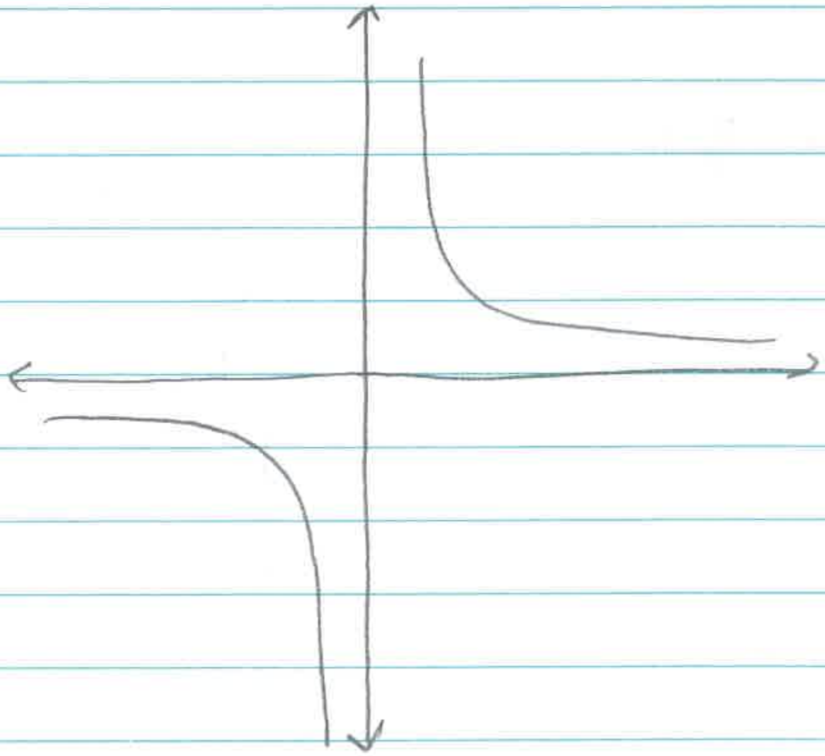
x	$y=\sqrt{x}$
0	0
1	1
2	$\sqrt{2}$
3	$\sqrt{3}$
4	2



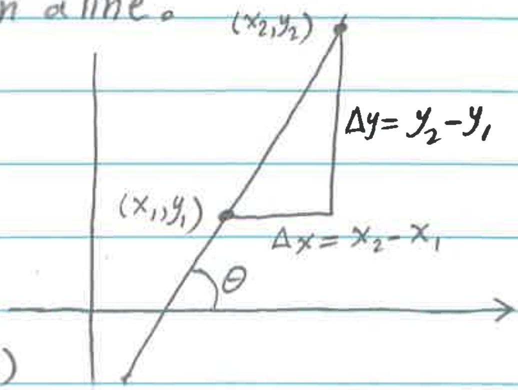
EX1 Sketch the graph $y=\frac{1}{x}$

عدا الصفر

x	$y=\frac{1}{x}$
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
$-\frac{1}{3}$	-3
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-3	$-\frac{1}{3}$



Def:- The slope of the line is calculated by finding the ratio of the "vertical change" to the "horizontal change" between any two distinct points on a line.



$$m = \text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The equation of a line is $y - y_1 = m(x - x_1)$

Ex) In each part find the slope of the line through

(a) The points $(6,2)$ and $(9,8)$

Sol) $m = \frac{8-2}{9-6} = \frac{6}{3} = 2$

(b) The points $(2,9)$ and $(4,3)$

$m = \frac{3-9}{4-2} = \frac{-6}{2} = -3$

(c) The points $(-2,7)$ and $(5,7)$

Sol) $m = \frac{7-7}{5-(-2)} = 0$

Theorem :-

1- Two nonvertical lines with slopes m_1 and m_2 are paralled iff they have the same slope, that is $m_1 = m_2$.

2- Two nonvertical lines with slopes m_1 and m_2 are perpendicular iff the product of their slopes -1, that is

$$m_1 m_2 = -1 \Rightarrow m_1 = \frac{-1}{m_2} \Rightarrow m_2 = \frac{-1}{m_1}$$

3- The line passing through $P_1(x_1, y_1)$ and having slope m is given by the equation

$$y - y_1 = m(x - x_1)$$

4- The vertical line through $(a, 0)$ and the horizantal line through $(0, b)$ are represented respectively by the equations

$$x = a \text{ and } y = b$$

5- The line with y-interecpt b and slope m is given by the equation $y = mx + b$

Ex/ Find the point slope formula for the line through the point $(4, -3)$ with slope $m = 5$

Sol: $y + 3 = 5(x - 4)$

$y + 3 = 5x - 20$

$y - 5x + 23 = 0$

Ex1

Equations	slope (m)	
$y = 3x + 7$	$m = 3$	$b = 7$
$y = -x + \frac{1}{2}$	$m = -1$	$b = \frac{1}{2}$
$y = x + 1$	$m = 1$	$b = 1$
$y = \sqrt{2}x - 8$	$m = \sqrt{2}$	$b = -8$
$y = 2$	$m = 0$	$b = 2$

Ex1 Find the slope-intercept form of the equation of the line that satisfies the stated conditions:

(a) slope is -9 cross the y-axis at $(0, -4)$

Sol $m = -9$, $b = -4 \Rightarrow y = mx + b = -9x - 4$

(b) slope is 1, passes through the origin

Sol $m = 1$, $b = 0 \Rightarrow y = x + 0 \Rightarrow y = x$

(c) pass through $(5, -1)$, perpendicular to $y = 3x + 4$

Sol $y = 3x + 4 \Rightarrow m_1 = 3 \Rightarrow m_2 = \frac{-1}{3}$

$\Rightarrow y - y_1 = m(x - x_1) \Rightarrow y - (-1) = \frac{-1}{3}(x - 5)$

$\Rightarrow y = \frac{-1}{3}x + \frac{2}{3}$

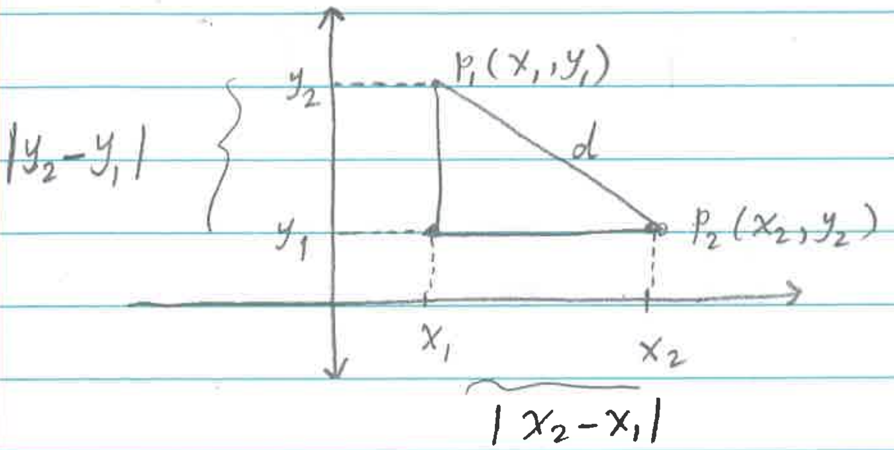
d) pass through $(3, 4)$ and $(2, -5)$

Sol $m = \frac{-5-4}{2-3} = 9$

Distance Equation

Suppose that we are interested in finding d between two points $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ in the xy-plan

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Ex/ Find the distance between the point $(\overset{x_1}{-2}, \overset{y_1}{3})$ and $(\overset{x_2}{1}, \overset{y_2}{7})$

Sol/ $d = \sqrt{(1-(-2))^2 + (7-3)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

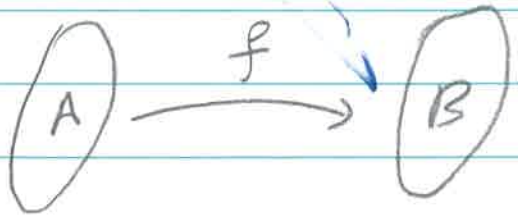
Thm/ The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) in coordinate plane is:

$$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right)$$

Ex/ Find the midpoint of the line segment joining $(3, -4)$ and $(7, 2)$

Sol/ $\left(\tfrac{1}{2}(3+7), \tfrac{1}{2}(-4+2)\right) = (5, -1)$

Def:- A function is a relation between two set A and B such that $\forall x \in A$, $\exists y \in B$, y is unique and define $f(x) = y$, $f : A \longrightarrow B$. we called A is the domain of f and B is the range of f.



Ex/ $y = x^2$ is a function with domain $= \mathbb{R}$ and Rang $= \mathbb{R}^+$

Ex/ $y = f(x) = x$ is a fun. with $D = \mathbb{R}$, $R_a = \mathbb{R}$

Def:- we take $y = f(x)$, this equation expresses y as a function. The variable x is called independ variable (or argument) of f, and the variable y is called the dependent variable of f.

Def: If $y = f(x)$, then the set of all possible input (x-value) is called the domain of f, and the set of outputs (y-value) that result when x varies over the domain is called the range of f.

Ex1 Find the domain and range of

ⓐ $f(x) = 2 + \sqrt{x-1}$

Sol $D = \{x \in \mathbb{R}, x \geq 1\}$, $R_a = \{y \in \mathbb{R}, y \geq 2\}$

b) $f(x) = (x+1)/(x-1)$

Sol $D = \{x \in \mathbb{R} ; x \neq 1\}$, $R_a = \{y \in \mathbb{R} ; y \neq 1\}$

c) $f(x) = \dfrac{1}{|x-2|}$

Sol $D = \{x \in \mathbb{R} ; x \neq 2\}$, $R_a =$

when $x = 2^+ \longrightarrow f(x) = \dfrac{1}{|2.1-2|} \longrightarrow 10$

biger & biger $\longrightarrow +\infty$

$x = 2^- \longrightarrow f(x) = \dfrac{1}{|1.9-2|} \longrightarrow 10$

$\longrightarrow +\infty$

$\therefore R = (0, +\infty)$.

Def: Given function f and g, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)\, g(x)$$
$$(f/g)(x) = f(x) / g(x)$$

Ex1 Let $f(x) = 1 + \sqrt{x-2}$ and $g(x) = x-3$, find:

1) $(f+g)(x) = f(x) + g(x) = (1+\sqrt{x-2}) + (x-3)$
$$= x - 2 + \sqrt{x-2}$$

$D = \{x \in \mathbb{R} ; x \geq 2\}$, $R_a = \{x \in \mathbb{R} ; x \geq 0\}$.

2) $(f-g)(x) = f(x) - g(x)$
$$= (1+\sqrt{x-2}) - (x-3) = 4 - x + \sqrt{x-2}$$

$D = \{x \in \mathbb{R} ; x \geq 2\}$

3) $(fg)(x) = f(x) \cdot g(x) = (1+\sqrt{x-2}) \cdot (x-3)$

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$D = \{x \in \mathbb{R} ; x \geq 2\}$, $R = \{x \in \mathbb{R} ; x \geq \frac{32}{27}\}$

4) $(f/g)(x) = f(x) / g(x) = (1+\sqrt{x-2}) / (x-3)$

$D = [2,3) \cup (3, \infty)$

5) $(7f)(x) = 7f(x) = 7 + 7\sqrt{x-2}$

$\Rightarrow D = \{x \in \mathbb{R} ; x \geq 2\}$.

Remark: The functions $f+g$, $f-g$, fg, we define the domain to be the intersection of the domain of f and g ; and the domain of f/g to be the intersection of the domain of f and g but without the points where $g(x)=0$ exclude.

Def: Given function f and g, the composition of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is defined to consist of all x in the domain of g and with $g(x)$ is in the domain of f

let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$ find

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

sol

$(f \circ g)(x) = f(g(x)) = [g(x)]^2 + 3 = (\sqrt{x})^2 + 3 = x + 3$

$D = [0, \infty)$

(b) $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{x^2+3}$

$D = (-\infty, \infty)$

Remark:- compositions can also be defined for three or more functions for example. $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)).$$

In other words, first find $h(x)$, then find $g(h(x))$ and then find $f(g(h(x))$.

Ex) Find $(f \circ g \circ h)(x)$ if $f(x) = \sqrt{x}$, $g(x) = \frac{1}{x}$, $h(x) = x^3$

Sol $(f \circ g \circ h)(x) = f(g(h(x))$

$$= f(g(x^3)) = f\left(\frac{1}{x^3}\right) = \sqrt{\frac{1}{x^3}} = \frac{1}{x^{\frac{3}{2}}}$$

Ex) Express $h(x) = (x-4)^5$ as a composition of two functions

$g(x) = x-4$, $f(x) = x^5 \Longrightarrow h = f(g(x))$.

Def:- A function is called even if $f(-x)=f(x)$ for example,

$f(x)=x^2 \Longrightarrow f(-2)=f(2)=4$

(2) A function is called odd if $f(-x)\neq f(x)$

Ex/ $f(x)=x \longrightarrow f(-2)=-2$ and $f(2)=2$

3) A function is called one to one; $f: A \longrightarrow B$ if $\forall\, y\in B,\ \exists\, x\in A$

$\Rightarrow f(x)=y$, for example $y=x$

4) A function is called onto; $f: A \longrightarrow B$, if $f(A)=B$

Ex/ $y=x$

5) A function $y=f(x); A \longrightarrow B$ is one to one and onto, if we have inverse function $x=f^{-1}(y); B\longrightarrow A$

Ex/ $y=f(x)=x^2$

$D=\mathbb{R}^+$, $R_a=\mathbb{R}^+$ is one to one, onto with the inverse function $x=\sqrt{y}$, $D=\mathbb{R}^+$, $\mathbb{R}_a=\mathbb{R}^+$.

EX1 Find $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1+x}\right)$

Sol $\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right)$$

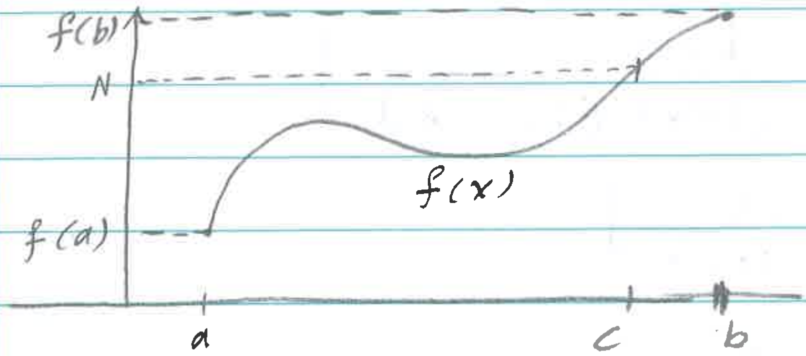
$$= \sin^{-1}\left[\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}}\right]$$

$$= \sin^{-1}\left[\lim_{x\to 1} \frac{\cancel{1-x}}{\cancel{(1-x)}\,(1+\sqrt{x})}\right]$$

$$= \sin^{-1}\left[\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right]$$

$$= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

The intermediate value Theorem: Suppose that f is
i.e for each value between the least upper bound and greatest lower bound of the image of cont fun. there is at least one point in its domain.
cont. on the closed interval $[a,b]$ and let N be any number between $f(a)$ & $f(b)$, where $f(a) \neq f(b)$. Then $\exists$ a number c in (a,b) such that $f(c)=N$



in order to be root we have to say equal 0 in some where

Ex1 Show that there is a root of the equation.

its cont since it is poly → $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 & 2

if we will try to evaluated at 1 and 2 and one of them is positive and the other one is negative then we would know that it had 0 by intermediate Thm.

$f(1) = 4-6+3-2 = -1 \quad < 0$

$f(2) = 32-24+6-2 = 12 \quad > 0$

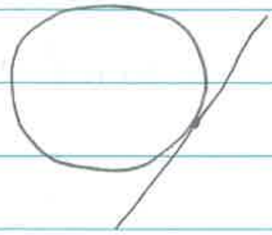
$\therefore$ by IVT there exist a number $c \in (1,2)$ such that $f(c) = 0$.

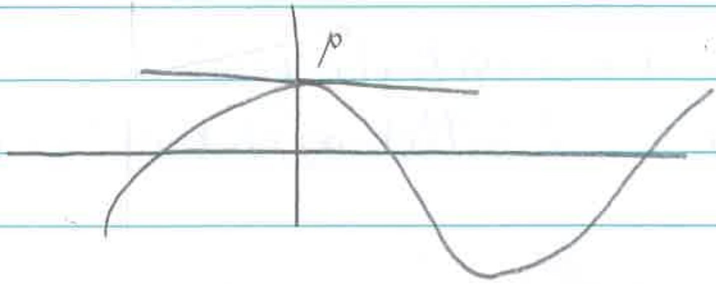
slope $= \frac{\text{change in } y}{\text{change in } x}$

Limits

First we can look of some examples of tangent curves and when I usually ask Q what is the tangent to the curve.

Tangent curves its a line hits a curve in one point $p = (x, f(x))$

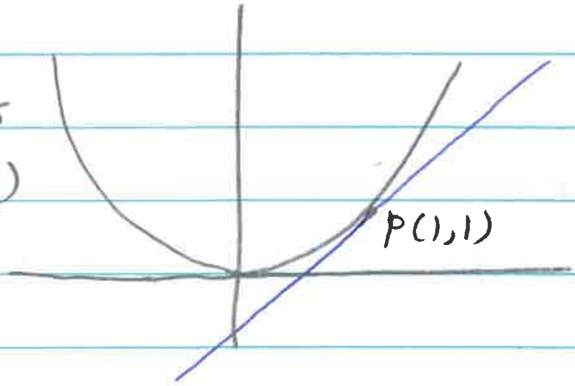




The problem is that we need two points to compute the slope.

EX1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $p(1,1)$.

we see that the tangent line is the blue line at the point (1,1) and this what we want to do to find the eq. of this line (how we do that)



how normaly find eq of the line ?

The problem we dont have two point on the tangent line.

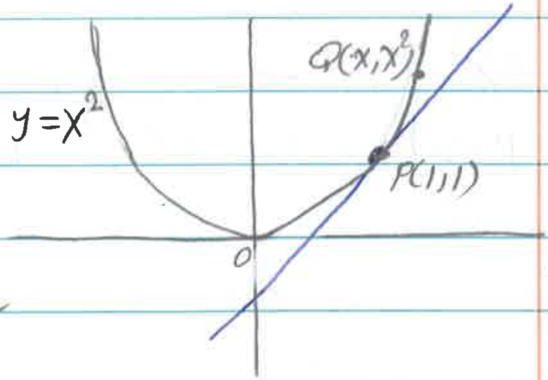
So What we do ?

we pick a nother point Q it welly welly close to the point p and use this point to approximate the equation of the tangent line.

Note: if you pick the point Q closer to the point p you will get better approximation.

So this our process it will be in this section.

EX/ Find an equation of the tangent line to the ~~prab~~ parabola
$y = x^2$ at the point $p(1,1)$.



Sol

we will be able to find an equation of the tangent line as soon as we know its slope m.
The difficulty is that we know only one point p, whereas we need two points to compute the slope.
But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola and computing the slope m_{PQ}

we choose $x \neq 1$ so that $Q \neq p$. then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25-1}{1.5-1} = \frac{1.25}{0.5} = 2.5$$

The tables show the values of m_{PQ} for several values of x close to 1.

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

we see when x is realy close to 1 we get it

The closer Q to P, the closer x is to 1 and, it appears the tables, the closer m_{PQ} is to 2.

This suggests that the slope of the tangent line should be $m=2$

i.e $\lim_{Q \to P} m_{PQ} = m$ and $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$

So we can find the equation of this line using

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x-1)$$

$$\Rightarrow y = 2x - 1$$

Ex1 Find the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

Sol: The function $f(x) = \frac{\sin x}{x}$ is not defined when $x = 0$

we constrate table of values correct to eight decimal place

x	$\frac{\sin x}{x}$
∓ 1.0	0.84147098
∓ 0.5	0.95885108
∓ 0.4	0.97354586
∓ 0.3	0.98506736
∓ 0.2	0.99334665
∓ 0.1	0.99833417
∓ 0.5	0.99958339
∓ 0.01	0.99998333
∓ 0.005	0.99999583
∓ 0.001	0.99999983

So $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Def :- suppose $f(x)$ is defined when x is near the number a, we say the ~~li~~ $\lim_{x \to a} f(x) = L$ (the limit of f(x), as x approches a equal L)

if we can make the values of f(x) close to L as x gets really close to a.

Note :- 1) The function does need to be defined at $x=a$
2) we must look as $x \to a$ from both ~~the~~ the left and right.

EX1 Find the $\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

what we know about this we the demenator $\neq 0$
Note: at $x=1$ the function f(x) is not defined

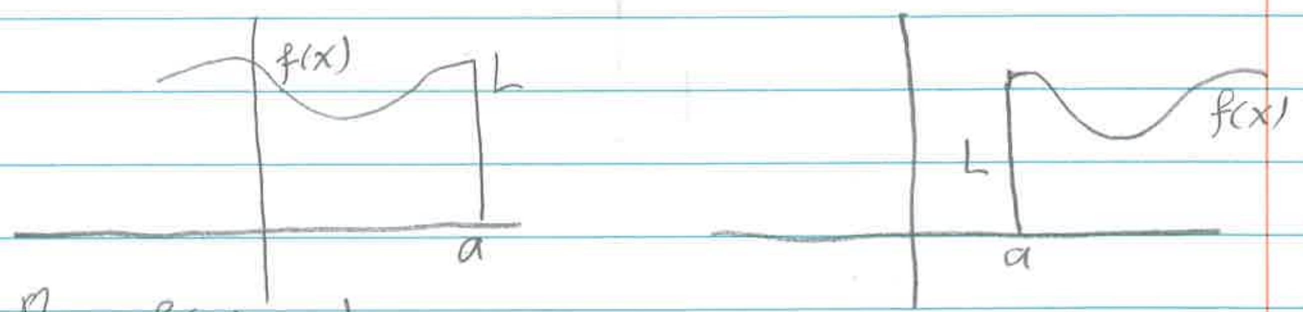
we need to look for both side from left and right and we see from the table

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
.9999	0.500025	1.0001	0.499975

EX! $\lim_{x \to 0} \frac{\sin x}{x} = 1$, $\lim_{x \to 0} \frac{\sin \pi}{x} = 0$

Def: We write $\lim_{x \to a^-} f(x) = L$ and we say the left-hand limit of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a from the left] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Similary, we get right-hand limit of $f(x)$ as x approches a is equal to L and we write $\lim_{x \to a^+} f(x) = L$.



$\lim_{x \to a^-} f(x) = L$ $\lim_{x \to a^+} f(x) = L$

Note: The relationship between limits and one-sided limits is given by the following:

$\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L$ & $\lim_{x \to a^+} f(x) = L$

Ex1 Show $\lim_{x \to 0} |x| = 0$

Sol.1 $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$

$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

Hence $\lim_{x \to 0} |x| = 0$ $\{\text{since } \lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0\}$.

Ex1 prove $\lim_{x \to 0} \frac{|x|}{x}$ does not exist

Sol.1 $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$

$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$

Since $\lim_{x \to 0^-} \frac{|x|}{x} \neq \lim_{x \to 0^+} \frac{|x|}{x}$

Hence $\lim_{x \to 0} \frac{|x|}{x}$ DNE

Infinite limits:- If the values of $f(x)$ increase indefinitely as x approaches a from the right or left, then we write $\lim_{x \to a^+} f(x) = +\infty$ or $\lim_{x \to a^-} f(x) = +\infty$

as approprite, we say that $f(x)$ increases without bound, or $f(x)$ approaches $+\infty$, as $x \longrightarrow a^+$ or as $x \longrightarrow a^-$

Similarly, if the values of $f(x)$ decrease indifinitely as x approaches a from the right or left, then we write $\lim_{x \to a^+} f(x) = -\infty$ or $\lim_{x \to a^-} f(x) = -\infty$

as appropriate, and say that $f(x)$ decreases without bound, or $f(x)$ approaches $-\infty$, as $x \longrightarrow a^+$ or as $x \longrightarrow a^-$, Moreover, if both one-sided limits are $+\infty$, then we write

$$\lim_{x \to a} f(x) = +\infty$$

and if both one-sided limits are $-\infty$, then we write

$$\lim_{x \to a} f(x) = -\infty$$

i.e we write symbolically $\lim_{x \to a} f(x) = \infty$ to indicate that the value tend to become larger and larger as x becomes closer and closer to a.

Ex1 Consider the function $g(x) = \dfrac{10x}{x-2}$ determine;

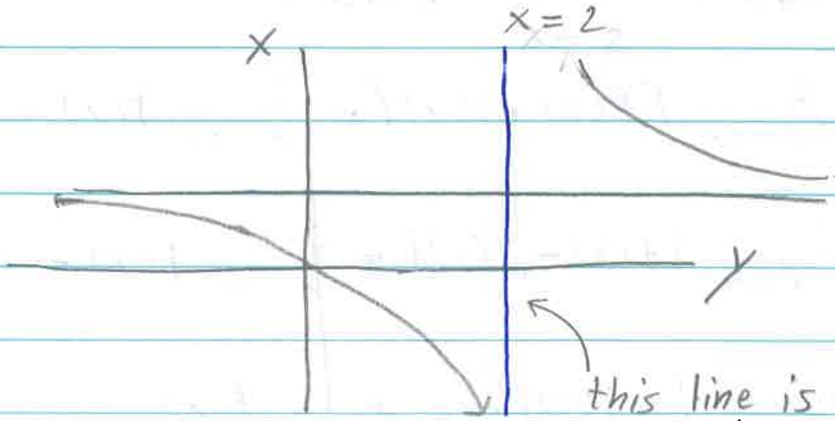
(a) $\lim\limits_{x \to 2^-} g(x)$

x	1	1.9	1.99	1.999	$\longrightarrow 2$
$g(x)$	$\frac{10}{-1} = -10$	-190	-1990	-19990	$\longrightarrow -\infty$

Hence $\lim\limits_{x \to 2^-} g(x) = -\infty$

(b) $\lim\limits_{x \to 2^+} g(x)$

x	3	2.1	2.01	2.001	$\longrightarrow 2$
$g(x)$	30	210	2010	20010	$\longrightarrow \infty$



this line is called vertical line

Def: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statement is true;

$\lim\limits_{x \to a} f(x) = \infty$ $\quad$ $\lim\limits_{x \to a^-} f(x) = \infty$ $\quad$ $\lim\limits_{x \to a^+} f(x) = \infty$

$\lim\limits_{x \to a} f(x) = -\infty$ $\quad$ $\lim\limits_{x \to a^-} f(x) = -\infty$ $\quad$ $\lim\limits_{x \to a^+} f(x) = -\infty$

__Thm1__ Let a and k be a real numbers

$$\lim_{x\to a} k = k \;,\quad \lim_{x\to a} x = a \;,\quad \lim_{x\to 0^-} \frac{1}{x} = -\infty ,\quad \lim_{x\to 0^+} \frac{1}{x} = +\infty$$

__Ex 1__ $\lim_{x\to\pi} 4 = 4$, $\lim_{x\to 0} 4 = 4$, $\lim_{x\to 4} 4 = 4$.

So far we evalating some limit by using calculator to guss the limits, now we will use some properties to find the limits

__limit laws:-__ Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$ exists then!

the limit of sum is the sum of the limit

1) $\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

the limit of diff is the diff of the limits

2) $\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

limit of a constant times a function is the constant times the limit of the function

3) $\lim_{x\to a} [c f(x)] = c \lim_{x\to a} f(x)$

the limit of the product is the product of the limit

4) $\lim_{x\to a} [f(x) g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

the limit of the quotient is the quotient of the limit

5) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

6) $\lim_{x\to a} x^n = (\lim_{x\to a} x)^n = a^n$

Ex/ Find $\lim_{x \to 5} (x^2 - 4x + 3)$

$= \lim_{x \to 5} x^2 - \lim_{x \to 5} 4x + \lim_{x \to 5} 3$

$= 5^2 - 20 + 3 = 8$

Ex/ $\lim_{x \to 2} \frac{5x^3 + 4}{x - 3}$

$= \frac{\lim_{x \to 2} (5x^3 + 4)}{\lim_{x \to 2} (x - 3)} = \frac{5 \cdot 2^3 + 4}{2 - 3} = -44$

More limit properties

1) $\lim_{x \to a} x = a$ Ex/ $\lim_{x \to 5} x = 5$

2) $\lim_{x \to a} x^n = a^n$ where n is positive integer
(if n is even, we assume that $a > 0$)

3) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is positive

[if n is even, we assume that $\lim_{x \to a} f(x) > 0$]

Ex/ (a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

$= \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$

$= 2(\lim_{x \to 5} x)^2 - 3 \lim_{x \to 5} x + 4$

$= 2(5)^2 - 3(5) + 4$

$= 39.$

(b) $\lim_{x \to -2} \dfrac{x^3+2x^2-1}{5-3x}$

$= \dfrac{\lim_{x \to -2} x^3+2x^2-1}{\lim_{x \to -2} 5-3x}$

$= \dfrac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \dfrac{-1}{11}$

- Functions with the direct substitution property are called continuous at a.

Remember! not all limits can be evaluated by direct subsitution. (because it has a hall)

Ex1 Find the $\lim_{x \to 1} \dfrac{x^2-1}{x-1}$

You notes right the way you can not use direct sum here because if you try to do direct substitution you will get 0 on the denomenator and we cant divid by Zero. So lets try factoring the top.

$= \lim_{x \to 1} \dfrac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \to 1} x+1 = 1+1 = 2.$

Ex1 Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

The problem is we cant use the direct substitusion because we have o in the denomenator. So we need to do something, let we try to rationaliz this nomerator

$$= \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} \cdot \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$= \lim_{t \to 0} \frac{(t^2+9)-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t \to 0} \frac{\cancel{t^2}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{3+3} = \frac{1}{6}$$

Ex1 $\lim_{x \to 5} \frac{x^2-3x-10}{x^2-10x+25} = \lim_{x \to 5} \frac{(x-5)(x+2)}{(x-5)(x-5)}$

$$= \lim_{x \to 5} \frac{x+2}{x-5} = \frac{7}{0} \text{ DNE}$$

Ex1 $\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1} = \lim_{x \to 0} \frac{x}{\sqrt{x+1}-1} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)}$

$$= \lim_{x \to 0} \frac{x(\sqrt{x+1}+1)}{(x+1)-1}, \quad x \neq 0$$

$$= \lim_{x \to 0} (\sqrt{x+1}+1)$$

$$= 2.$$

Note! $\lim\limits_{x \to +\infty} x = +\infty$, $\lim\limits_{x \to -\infty} x = -\infty$

$\lim\limits_{x \to -\infty} \frac{1}{x} = 0$, $\lim\limits_{x \to +\infty} \frac{1}{x} = 0$.

Ex! $\lim\limits_{x \to +\infty} \frac{1}{x^4} = 0$ and $\lim\limits_{x \to -\infty} \frac{1}{x^4} = 0$

Note! $\lim\limits_{x \to -\infty} x^n = \begin{cases} -\infty & n = 1, 3, 5, 7, \dots \\ +\infty & n = 2, 4, 6, \dots \end{cases}$

Ex! $\lim\limits_{x \to +\infty} 2x^5 = +\infty$, $\lim\limits_{x \to +\infty} -7x^6 = -\infty$

$\lim\limits_{x \to -\infty} 2x^5 = -\infty$; $\lim\limits_{x \to -\infty} -7x^6 = -\infty$

Ex! Find $\lim\limits_{x \to +\infty} \frac{3x+5}{6x-8}$

Sol $\lim\limits_{x \to +\infty} \frac{3x+5}{6x-8} = \lim\limits_{x \to +\infty} \frac{x(3+\frac{5}{x})}{x(6-8/x)}$

$= \lim\limits_{x \to +\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}$

$= \frac{3+0}{6-0} = \frac{1}{2}$

limits of polynomials as $x \to \mp\infty$

The end behavior of a poly matches the end behavior of its highest degree term. More precisely if $c_n \neq 0$ then

$$\lim_{x\to\mp\infty}(c_0+c_1x+\cdots+c_nx^n)=\lim_{x\to\mp\infty}c_nx^n$$

Ex1 $\displaystyle\lim_{x\to-\infty}(-4x^8+17x^3-5x+1)=\lim_{x\to-\infty}-4x^8=-\infty$

Ex1 $\displaystyle\lim_{x\to+\infty}\frac{3x^3-2x^2+1}{3x+5}$

Sol $\displaystyle\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{3x+5}=\lim_{x\to+\infty}\frac{5x^2-x-\frac{1}{x}}{3+\frac{5}{x}}=+\infty$

Ex1 $\displaystyle\lim_{x\to+\infty}\sqrt[3]{\frac{3x+5}{6x-8}}$

Sol $\displaystyle\cancel{\lim_{x\to+\infty}}\sqrt[3]{\lim_{x\to+\infty}\frac{3x+5}{6x-8}}=\sqrt[3]{\frac{1}{2}}$

Ex1 $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{x^2+2}}{3x-6}$

$$=\lim_{x\to+\infty}\frac{\sqrt{\frac{x^2+2}{x^2}}}{\frac{(3x-6)}{x}}=\frac{\lim_{x\to+\infty}\sqrt{1+\frac{2}{x^2}}}{\lim_{x\to+\infty}\left(3-\frac{6}{x}\right)}$$

$$=\frac{\sqrt{\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)}}{\lim_{x\to+\infty}\left(3-\frac{6}{x}\right)}=\frac{\sqrt{1+0}}{3-0}=\frac{1}{3}$$

Ex1 $\lim\limits_{x \to +\infty} \left(\sqrt{x^6+5} - x^3\right)$

$$= \lim_{x \to +\infty} \left(\sqrt{x^6+5} - x^3\right) \cdot \left(\frac{\sqrt{x^6+5} + x^3}{\sqrt{x^6+5} + x^3}\right)$$

$$= \lim_{x \to +\infty} \frac{(x^6+5) - x^6}{\sqrt{x^6+5} + x^3} = \lim_{x \to +\infty} \frac{5}{\sqrt{x^6+5} + x^3}$$

$$= \lim_{x \to +\infty} \frac{\frac{5}{x^3}}{\sqrt{1 + \frac{5}{x^0}} + x^3}$$

$$= \frac{0}{\sqrt{1+0} + 1} = 0$$

Ex1 $\lim\limits_{x \to +\infty} \left(\sqrt{x^6+5x^3} - x^3\right)$

$$= \lim_{x \to +\infty} \left(\sqrt{x^6+5x^3} - x^3\right) \cdot \left(\frac{\sqrt{x^6+5x^3} + x^3}{\sqrt{x^6+5x^3} + x^3}\right)$$

$$= \lim_{x \to +\infty} \frac{(x^6+5x^3) - x^6}{\sqrt{x^6+5x^3} + x^3}$$

$$= \lim_{x \to +\infty} \frac{5x^3}{\sqrt{x^6+5x^3} + x^3}$$

$$= \lim_{x \to \infty} \frac{5x^3}{\sqrt{\frac{x^6}{x^6} + \frac{5x^3}{x^6}} + \frac{x^3}{x^6}}$$

$$= \lim_{x \to \infty} \frac{5}{\sqrt{1 + \frac{5}{x^3}} + 1} \neq \frac{5}{\sqrt{1+0} + 1}$$

$$= \frac{5}{2}$$

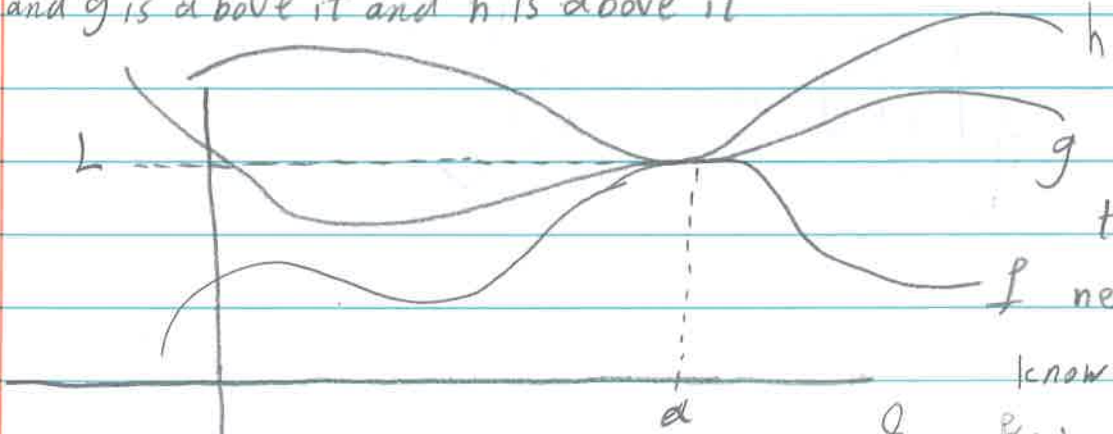
Thm:- if $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f & g both exist as x approaches a, then $\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$.

This lead to Squeeze Thm which is a very important Thm.

Thm: The Squeeze Thm: if $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \text{ then } \lim_{x \to a} g(x) = L.$$

So we have the following nice picture, f is the lowest one and g is above it and h is above it



and let look to some value near a and we know

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

we say at a, $\lim_{x \to a} g(x) = L$

this what called the squeez Thm

Ex) Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

Sol we know $-1 \leq \sin\frac{1}{x} \leq 1$ {since we know x^2 is positive so dont need to worry}

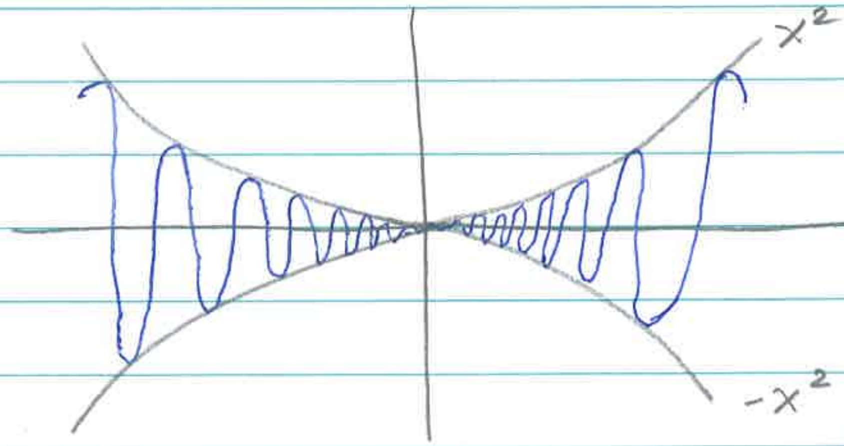
$\lim_{x \to 0} -x^2 = -(\lim_{x \to 0} x^2) = 0$

[...]et we do the same for upper one $\lim_{x \to 0} x^2 = (\lim_{x \to 0} x^2) = 0^2 = 0$

$\lim_{x \to 0} x^2 = (\lim_{x \to 0} x)^2 = 0^2 = 0$

$\therefore$ Since $\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0$

by The squeeze Thm it follows $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$



H.w: Find $\lim_{x} x^2 e^{\sin(\frac{1}{x})} = 0$

Sol $-1 \leq \sin\frac{1}{x} \leq 1$

take e raised to both sides of an inequality

$\Rightarrow e^{-1} \leq e^{\sin\frac{1}{x}} \leq e^1$

mult by x^2

$\Rightarrow x^2 e^{-1} \leq x^2 e^{\sin(\frac{1}{x})} \leq x^2 e^1$

The precise Definition of a limit

Def:- The distance between two real numbers a and b is

$$|a-b| = |b-a|$$

Question! In order to say that $\lim\limits_{x\to a} f(x) = L$. we must have values of $f(x)$ are arbitrarily close to L by taking values of x suffiently close to a but not equal to a. We can measure this "closness" by considering the quatities $|x-a|$ and $|f(x)-L|$ where $x \neq a$.
So how small does $|x-a|$ need to be in order for

EX! if $f(x) = \begin{cases} 2x-1 & \text{if } x\neq 3 \\ -4 & \text{if } x=3 \end{cases}$

Sol when x close to 3 but $x \neq 3$ then $\lim\limits_{x\to 3} f(x) = 3$
i.e $f(x)$ is close to 5

Q now how close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.1 ?

remember the distance from x to 3 is $|x-3|$ and the distance from $f(x)$ to 5 is $|f(x)-5|$.

So our problem to find δ st

$|f(x)-5| < 0.1$ if $|x-3| < \delta$ but $x \neq 3$

Then $|f(x)-5| = |(2x-1)-5| = |2x-6|$

$$= 2|x-3| < 2\ (0.05) = 0.1$$

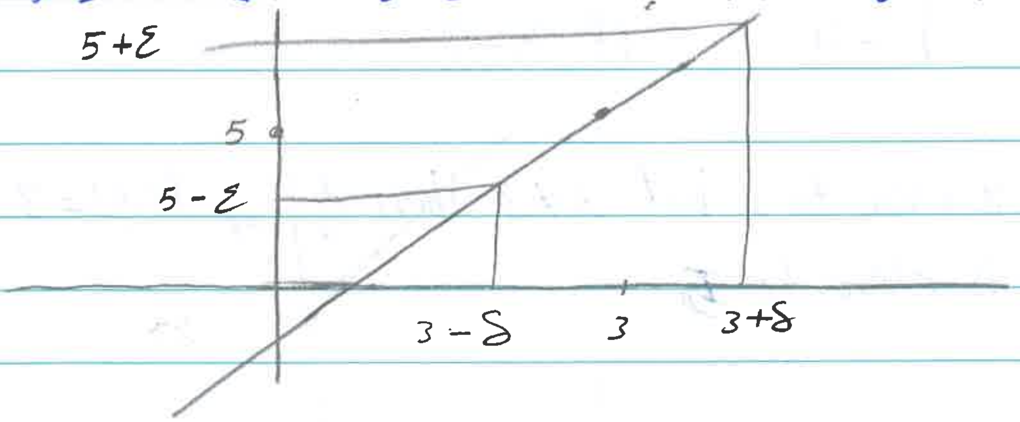
So, if $0 < |x-3| < \underset{\delta}{0.05}$ then $|f(x)-5| < 0.1$

So if x withen distance of 0.05 from 3, then $f(x)$ will be withen distance of 0.1 from 5

In more general

$|f(x)-5| < \varepsilon$ if $0 < |x-3| < \delta = \frac{\varepsilon}{2}$ ---- (1)

i.e if $3-\delta < x < 3+\delta$ $(x \neq 3)$ then $5-\varepsilon < f(x) < 5+\varepsilon$



Defi Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approches a is L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

if $0 < |x-a| < \delta$ then $|f(x)-L| < \varepsilon$.

Ex1 prove that $\lim_{x \to 3} (4x-5) = 7$

Sol let ε be a given positive number

we want to find a number δ such that

if $0 < |x-3| < \delta$ then $|(4x-5) - 7| < \varepsilon$

note, $|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3|$

So we want δ s.t if $0 < |x-3| < \delta$ then $4|x-3| < \varepsilon$ then
$|x-3| < \frac{\varepsilon}{4}$

so $\delta = \frac{\varepsilon}{4}$

2) To show δ its work

given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4}$

if $0 < |x-3| < \delta$ then

$|(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$

So, if $0 < |x-3| < \delta$ then $|(4x-5)-7| < \varepsilon$

Hence $\lim_{x \to 3} (4x-5) = 7$

the same thing we can give difinition in a precise way for infinite limite.

Def: Let f be a function on some open interval that contains the number a, except possibly at a itself. Then $\lim_{x\to a} f(x)=\infty$ means that for every positive number M and a positive number δ such that if $0<|x-a|<\delta$ then $f(x)>M$

same for $\lim_{x\to a} f(x)=-\infty$ but $f(x)<M$

Ex) prove that $\lim_{x\to 0} \frac{1}{x^2}=\infty$

Sol) Let M be a positive number.
we want to find a number δ s.t

if $0<|x|<\delta$ then $\frac{1}{x^2}>M$

but $\frac{1}{x^2}>M \Leftrightarrow x^2<\frac{1}{M} \Leftrightarrow |x|<\frac{1}{\sqrt{M}}$

So, if we choose $\delta=\frac{1}{\sqrt{M}}$ then $\frac{1}{x^2}>M$.

Ex) $\lim_{x\to 2^+} \frac{1}{x-2}=\infty$

let $M>0$, we want to find number δ s.t if
$|x-2|<\delta$ then $\frac{1}{x-2}>M$

note $\frac{1}{x-2}>M \Rightarrow x-2<\frac{1}{M}$, choose $\delta=\frac{1}{M}$.